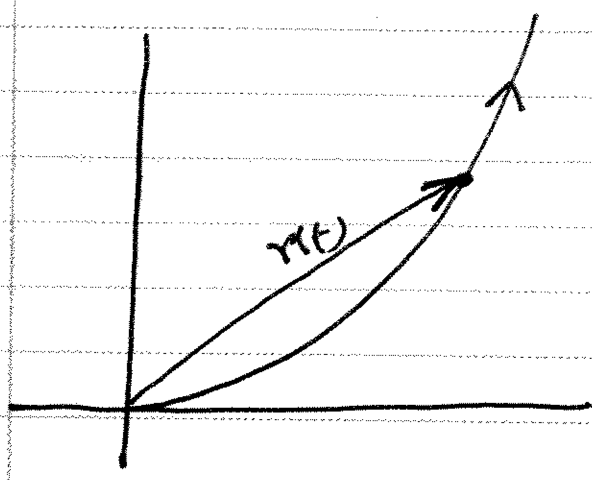
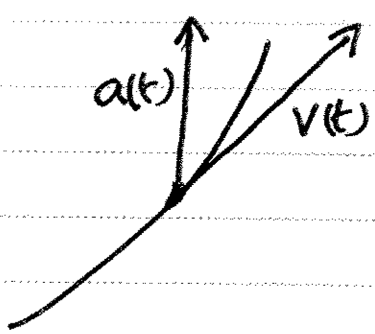
Let us look at the problem in Home Work 1.



$$r(t) = \begin{pmatrix} t \\ t^2 \end{pmatrix}$$



$$v(t) = \begin{pmatrix} 1 \\ 2t \end{pmatrix}$$

$$a(t) = \begin{pmatrix} 0 \\ 2 \end{pmatrix}$$

A point is moving along a parabolic trajectory.

Define tangential component of the acceleration

$$a_T(t) = \mathrm{proj}_{[v(t)]}\, a(t)$$

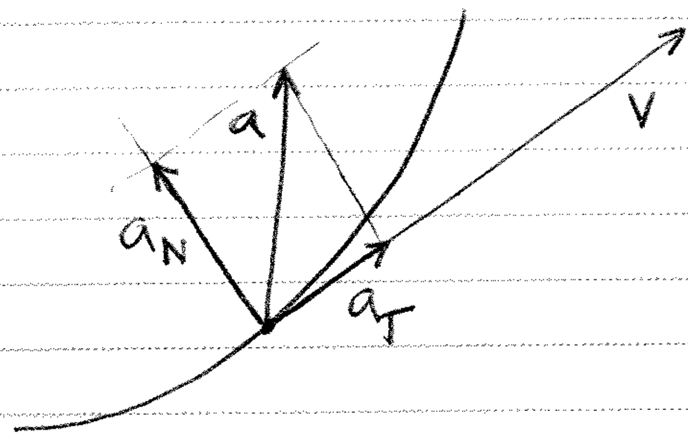
$$= \begin{pmatrix} 4t/(1+4t^2) \\ 8t^2/(1+4t^2) \end{pmatrix}$$

you need to calculate this

Use $a_T = \dfrac{a \cdot v}{v \cdot v}\, v(t)$

The normal component a_N is given by

$$a_N = a - a_T = \begin{matrix} -4t/(1+4t^2) \\ 2/(1+4t^2) \end{matrix}$$



Many often, one defines unit vectors along tangent and normal directions as follows:

$$u(t) = \frac{v(t)}{\|v(t)\|}$$

(unit tangent vector).

$$= \begin{pmatrix} \dfrac{1}{\sqrt{1+4t^2}} \\ \\ \dfrac{2t}{\sqrt{1+4t^2}} \end{pmatrix}$$

It turns out that $\dot{u}(t)$ is automatically perpendicular to $u(t)$ and is therefore in the normal direction.

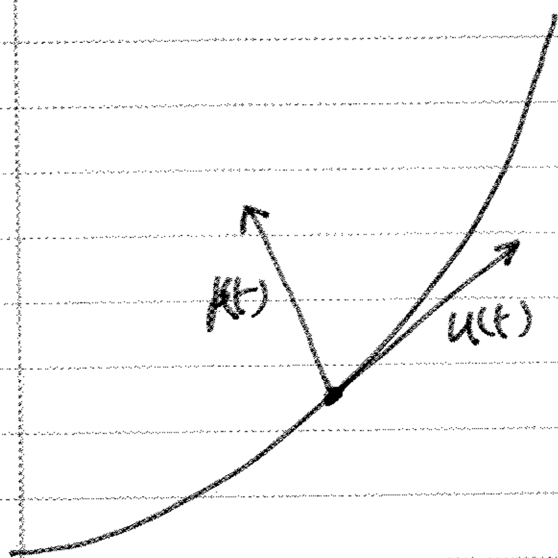
$$\dot{u}(t) = \frac{2}{(1+4t^2)^{3/2}} \begin{pmatrix} -2t \\ 1 \end{pmatrix}$$

which is perpendicular to u.
It is not of unit length though.

We can define

$$p(t) = \frac{\dot{u}}{\|\dot{u}\|} = \frac{1}{\sqrt{1+4t^2}} \begin{pmatrix} -2t \\ 1 \end{pmatrix}$$

$\boxed{\text{unit normal vector.}}$



Thus we have a unit tangent vector $u(t)$ & unit normal vector $p(t)$ associated with a curve on a plane.

Note that we also have

$$\dot{u}(t) = \frac{2}{1+4t^2}\, p(t)$$

(from page 4).

Can also calculate

$$\dot{p}(t) = -\frac{2}{1+4t^2}\, u(t)$$

The quantity

$$\frac{2}{1+4t^2}.$$

Correction:
Curvature is defined by normalizing this function \kappa(t) by the speed of the particle at t.
So Curvature=2/[(1+4t^2)^(3/2)]
This way, the defined curvature is independent of the speed and depends only on the shape of the trajectory.

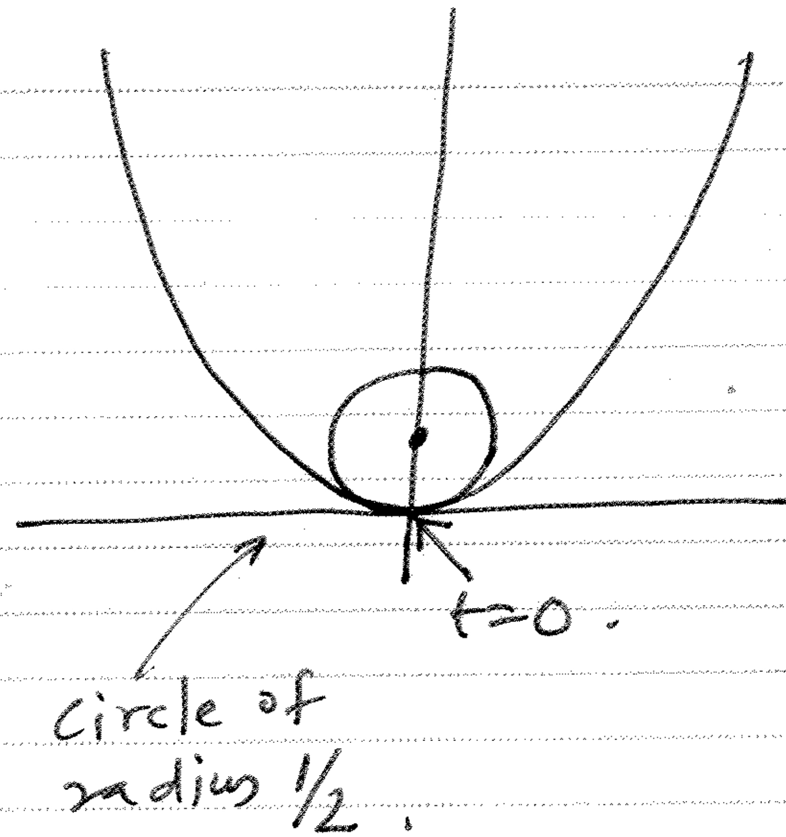
is denoted by $\mathcal{K}$ and is called the "curvature," and we have

$$\dot{u}(t) = \mathcal{K}(t)\,p(t)$$
$$\dot{p}(t) = -\mathcal{K}(t)\,u(t) \qquad **$$

For any plane curve there is an eqn like ** that relates unit tangent with unit normal vector.

The curvature κ describes the shape of the curve at any pt. on the curve.

- Note that for t large $\kappa(t) = \frac{2}{1+4t^2}$ is small and approaches 0. This means that for large t, the parabola looks like a straight line
- At $t=0$ $\kappa = 2$. We define $1/\kappa$ to be the radius of curvature R. $R = 1/2$ for $t=0$. This means that the parabola is closely approximated by a ~~circ~~ circle of radius $1/2$.

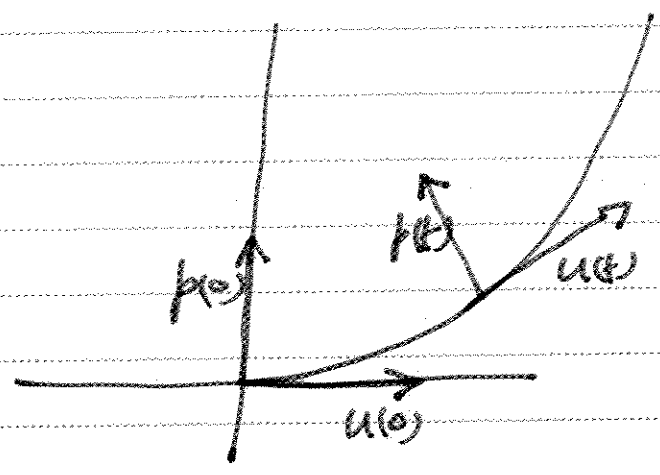

t=0.
circle of radius 1/2.

The differential equation

$$\dot{u} = \kappa p \qquad u(0) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$\dot{p} = -\kappa u \qquad p(0) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$\kappa = \frac{2}{1+4t^2}$$



describes how the co-ordinate axis moves with the curve.

How the co-ordinate axis moves is dictated precisely by the "shape" of the curve.

If you are looking for addibional challenges solve

$$\dot{u} = \kappa p \qquad \kappa = \frac{2}{1+4t^2}$$

$$\dot{p} = -\kappa u$$

$$\dot{\tilde{r}} = u$$

and solve the diff eq.

Take $\tilde{r}(0) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ $\quad u(0) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

$p(0) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

See that you get a parabola.
Change the i.c. and see that you still get a parabola oriented differently.

upshot:—

There is something magical about

$$K = \frac{2}{1+4t^2}$$

Independent of the initial condition, it defines the shape of a parabola through a diff. eq^n.

Remark:

In standard text, curvature is defined by normalizing the function \kappa by the speed. This normalized quantity is defined to be the curvature \kappa.

Thus we have

$$K = \frac{2}{1+4t^2}$$

$$\text{Curvature} = \frac{2}{(1+4t^2)^{3/2}}$$